Probability Density and current density

Let us check wether the Dirac equation leads to the correct probability density

The Dirac equation for a free particle.

$$[E - c\vec{\alpha}\cdot P - \beta mc^2]\psi = 0$$

where E and P are operators.

$$E \to i\hbar \frac{\partial}{\partial t}$$

$$P \to -i\hbar \nabla$$

$$i\hbar \frac{\partial \psi}{\partial t} + i\hbar c\vec{\alpha}\cdot\nabla\psi - \beta mc^2\psi = 0 \quad -(1)$$

A Hermitian conjugate equation given as.

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} - i\hbar c\nabla\psi^\dagger\cdot\vec{\alpha} - \psi^\dagger\beta mc^2 = 0 \quad -(2)$$

$\vec{\alpha}$ and β are Hermitian multiply (1) on the left by $\psi^\dagger$ and (2) on the right by ψ

$$i\hbar\psi^\dagger\frac{\partial\psi}{\partial t} + i\hbar c\psi^\dagger\alpha\cdot\nabla\psi - mc^2\psi^\dagger\beta\psi = 0$$

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t}\psi - i\hbar c\nabla\psi^\dagger\cdot\alpha\psi - mc^2\psi^\dagger\beta\psi = 0$$

after substracting

$$i\hbar\frac{\partial}{\partial t}(\psi^\dagger\psi) + i\hbar c\nabla\cdot(\psi^\dagger\vec{\alpha}\psi) = 0$$

We can thus identify the probability density and current density

$$\rho(r,t) = \psi^{\dagger}\psi$$

$$S(r,t) = c\psi^{\dagger}\alpha\psi$$

The Probability density is similar, if we note that $c\vec{\alpha}$ is the velocity of the particle.

$$i\hbar \frac{\partial x}{\partial t} = [x, H] = [x, c\vec{\alpha}.p + \beta mc^2] = i\hbar c\vec{\alpha}$$

$$\frac{\partial x}{\partial t} = c\vec{\alpha}$$

It is quite correct, because the position operator and the velocity operator have to define properly

The eigen values of the velocity operator are $c\vec{\alpha}$ this result is often attribute to Zitterbewegung and interpreted by uncertainity ~~prin~~ principle.